between 6 and 8. On the other hand the last two arrangements are different from the former and are different from each other. The order has been essentially disturbed and has been replaced by a different order which for some problems in the theory of numbers may be more convenient but which nobody would take to be the natural order. If we look a little more carefully we see that the last arrangement does not differ so radically from the third as at first appears to be the case. The order in the two has merely been reversed. The words "lie between" have the same validity in both but the meaning of "follow" and "precede" has been interchanged.

These observations lead to the idea that when a set of elements are arranged in a closed series, two arrangements in which the words "lie between," "follow," and "precede" have the same applicability may be called the same arrangement; two arrangements in which "lie between" preserves its characteristics, and "follow" and "precede" are interchanged may be called opposite; and two arrangements in which all three of the words change their significance when applied to the individual elements may be considered as essentially distinct. If there be given two sets of elements

0 1 2 3 4 5 6 7 8 9 0

and

A B C D E F G H I J A

which may be paired off in such a way that to each element of either set there corresponds one element of the other and only one and if each set possesses a definite arrangement of its members and if furthermore the words "follow," "precede," and "lie between" have the same significance when applied to the different members of each pair then the two sets of elements may be said to have the same arrangement.* Such is evidently the case in the example cited above.

When there is given a set of elements such as

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which do not possess any natural order or of which the natural order is not known, an order may be attributed to them arbitrarily by pairing them off against a set of elements which are arranged in order and by bodily transferring this order to them. Such a proceedure is often useful but it must be remembered that no real or natural order has thereby been obtained among the elements considered unless it can be shown that every such method of establishing an order leads to the same final result. For instance, let P be any point on the spherical surface. Describe any line which does not pass through the point P. Each di-

*The fact each of two sets of elements must possess its own independent characteristics if any valuable conclusions are to be drawn from establishing a correspondence between the sets has been overlooked by so eminent a mathematician as Hilbert. See my paper on The So-called Foundations of Geometry, *Archiv der Mathematik und Physik*, Vol. VI, pp. 104-122, 1903. It must however be evident that if the elements of a set B have no other properties than those derived from a one to one correspondence with the elements of a set A, then the set B is trivial and any thoorem obtained concerning its elements is really a theorem concerning the elements of the original set A. For this reason logicians are careful to compare not only the members of two sets but the operations and conceptions connected with them.

rection a issuing from the point P will cut this line in one point, say in A. The points A are arranged in a definite order upon the line and we may say that the directions a are arranged in the same order. We may say that a direction a' follows the direction a when the corresponding point A' follows the point A. The order which is thus attributed to the directions a is purely relative to and dependent on the order of the points on the particular line which has been chosen. Had some other line been fixed upon and the same process carried out there might conceivably have resulted a wholly different arrangement of the directions issuing from the point P. And until it has been proved that no matter what line was selected the same arrangement of the directions must result, we are in no position to affirm that the order found is at all a natural order.

To bring out even more clearly some of the ideas connected with order consider the series of positive integers and the series of all positive rational numbers. Each of these series possesses an obvious natural order; yet by rearranging the latter into an order wholly artificial it becomes possible to pair off the elements of the two series in a one to one manner. Any rational number may be written in the form p/q where p and q have no common factor. Arrange the rational numbers so that those in which the sum of numerator and denominator is smaller precede those in which the sum of numerator and denominator is greater and in case the sums are the same place the numbers which have the smaller numerator first. There results the series

$$1\ \ 2\ \ \tfrac{1}{2}\ \ 3\ \ \tfrac{1}{3}\ \ 4\ \ \tfrac{1}{4}\ \ \tfrac{2}{3}\ \ \tfrac{3}{2}\ \ 5\ \ \tfrac{1}{5}\ \ 6\ \ \tfrac{1}{6}\ \ \tfrac{2}{5}\ \ \tfrac{3}{4}\ \ \tfrac{4}{3}\ \ \tfrac{5}{2}\ \ 7\ \ \ldots\ldots$$

which contains once and only once every rational number and which may be put into one to one correspondence with the series of integers

$$1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ \ 11\ \ 12\ \ 13\ \ 14\ \ 15\ \ 16\ \ 17\ \ 18\ \ \ldots\ldots$$

Thus by sacrificing the natural order we may set up a one to one correspondence between these two series. It has been shown by somewhat complicated analysis that the points upon the segment of a line may be paired off against the points in a square. In this manner the points in a square may be considered to be ordered into a simple series in which the words "lie between," "follow" and "precede" are applicable. As however this order is wholly artificial and may be differently established in a great number of ways is is of very limited value.

Considerations like these shed a great light on the magnitude and significance of the assumptions at the bottom of the third axiom which states that the points of a line are arranged in a natural order. The fact of the arrangement alone is not of great import because almost any set of elements may be assigned an order. The word *natural* is what gives the axiom its significance. It states that the points of a line are distributed in an order which cannot be mistaken for nor confused with any other. If there were a number of equally natural orders which could not be at all times readily distinguished from one another the whole theory of segments would become unsettled. It also becomes clear that before

speaking of a natural order among the directions issuing from a point so many difficulties are to be met that a development of the theory of angle along lines similar to those pursued in discussing segments had best not be attempted any earlier than necessary.

Definition of (proper) triangle: *A (proper) triangle is the geometric figure composed of three segments each of which is less than a semi-line drawn so as to connect in pairs three points.* The parts of the triangle are its three sides, and its three angles which are the angles formed by the directions in which the segments leave the vertices. A segment less than a semi-line may be called a proper segment.

Theorem 10. *If two sides and the angle formed by them in one triangle are congruent to two sides and the angle formed by them in another triangle, the triangles are congruent.*

As two angles are congruent the triangles may be moved so that these angles coincide. The adjacent sides take the same directions and as they are congruent they must coincide throughout. The third sides will therefore lie on the same line and between the same points of the line. Of the two segments which satisfy these conditions one is greater than a semi-line and cannot form the side of a proper triangle. Hence the other segment must be the third side of each triangle and the triangles coincide throughout.

Theorem 11. *If a side and two adjacent angles of one triangle are congruent to a side and two adjacent angles of a second triangle, the triangles are congruent.*

Theorem 9 is needed in the proof—which is left to the reader. A similar proof may be given for the following:

Theorem 12. *If two angles are congruent their vertical angles are also congruent.*

Theorem 13. *If three directions a, b, c radiate from a point and three directions a', b', c' from the same or a different point and if furthermore the congruences $\measuredangle ab \equiv \measuredangle a'b'$ and $\measuredangle ac \equiv \measuredangle a'c'$ are fulfilled, then $\measuredangle bc \equiv \measuredangle b'c'$.*

ON SOME SPECIAL ARITHMETIC CONGRUENCES.

By H. S. VANDIVER, Bala, Pa.

If we take the well known relation

$$\left(\frac{p-1}{2}!\right)^2 \equiv (-1)^{(p+1)/2} (\mathrm{mod}\ p),$$

where p is a prime, and suppose that $p \equiv 3 (\mathrm{mod}\, 4)$, then

$$\left(\frac{p-1}{2}!\right)^2 \equiv 1 (\mathrm{mod} p) \ldots\ldots (1),$$

$$\frac{p-1}{2}! \equiv \pm 1 \pmod{p}, \ldots (2).$$

The question is, what analytic relations govern this ambiguous residue? Dirichlet (*Werke*, Vol. I, p. 107) called attention to the problem, and Jacobi (*Crelle*, Vol. 9, p. 189) enunciated and proved a theorem connected with it. Kronecker (Liouville, Vol. 3, 2nd series, p. 269) gave another theorem which may be used as a practical rule for determining the sign in any particular case. I wish to discuss here some relations connected with Jacobi's theorem, which is as follows:

If p is a prime number of the form $4n+3$, then

$$\frac{p-1}{2}! \equiv (-1)^{\mu} \pmod{p}$$

where μ is the number of quadratic non-residues of p which are less than $\frac{1}{2}p$.

Jacobi's proof of this depends on the theory of quadratic forms. Another proof is easily obtained from the formula

$$\left(\frac{p-1}{2}!\right)^{(p-1)/2} \equiv \frac{p-1}{2}! \pmod{p},$$

the truth of which is evident from (2) (see Bachmann, *Niedere Zahlentheorie*, Vol. I, pp. 178-9). The theorem shows that the least residue of $\frac{p-1}{2}! \pmod{p}$ is known as soon as the evenness or oddness of μ is determined. There are a number of other questions in the theory of numbers which involve a consideration of this same criterion, and some of them will be investigated here. It is necessary to make use of the analysis in another demonstration of Jacobi's theorem.

Consider the series

$$1^2, 2^2, 3^2, \ldots, \left(\frac{p-1}{2}\right)^2 \ldots (3),$$

and calculate the residues of these numbers, modulo p, between the limits $\frac{1}{2}(p-1)$ and $-\frac{1}{2}(p-1)$. No two of these residues can be equal in absolute value. For, let α^2 and β^2 be two terms of (3) which give equal remainders, irrespective of sign. Then

$$\alpha^2 \equiv \pm \beta^2 \pmod{p}.$$

Now we cannot have $\alpha^2 - \beta^2 \equiv 0 \pmod{p}$ since neither $\alpha - \beta$ nor $\alpha + \beta$ is zero, and each is less than p. The relation $\alpha^2 + \beta^2 \equiv 0 \pmod{p}$ is also impossible since p has the form $3 \pmod 4$. Hence the absolute values of the residues are identical, in some order, with the integers

$$1, 2, 3, \ldots, \tfrac{1}{2}(p-1).$$

The positive residues in question constitute all the quadratic residues of p which are less than $\frac{1}{2}p$. Consequently there will be precisely as many negative residues as there are quadratic non-residues less than $\frac{1}{2}p$. By multiplication we then obtain

$$\left(\frac{p-1}{2}!\right)^2 \equiv (-1)^{\mu}\left(\frac{p-1}{2}!\right) (\mathrm{mod}\, p),$$

from which the theorem follows, $\frac{p-1}{2}!$ being prime to p. This proof is of some interest in itself, since no use is made of Wilson's theorem.

Let $-a_1, -a_2, \ldots, -a_\mu$, be the negative least residues, modulo p, for the numbers in the series (3), and $\beta_1, \beta_2, \ldots, \beta_k$, the positive residues. Now write

$$1^2 = [1^2/p]p + r_1, \quad 2^2 = [2^2/p]p + r_2, \ldots, (p')^2 = [(p')^2/p]p + r_p,$$

where $[m/n]$ represents the largest integer in m/n, and $p' = \frac{1}{2}(p-1)$. Hence $r_1, r_2, \ldots, r_{p'}$ are the least *positive* residues for the series (3), modulo p. They are therefore identical apart from their order with $p-a_1, p-a_2, \ldots, p-a_\mu$, $\beta_1, \beta_2, \ldots, \beta_k$. Hence by addition,

$$1^2 + 2^2 + 3^2 + \ldots + (p')^2 = Mp + \mu p + B - A,$$

$$\text{or } \tfrac{1}{24}p(p-1)(p+1) = Mp + \mu p + B - A \ldots\ldots (4),$$

where

$$M = [1^2/p] + [2^2/p] + \ldots + [(p')^2/p], \quad A = a_1 + a_2 + \ldots + a_\mu, \quad B = \beta_1 + \beta_2 + \ldots + \beta_k.$$

Since $a_1, a_2, \ldots, a_\mu, \beta_1, \beta_2, \ldots, \beta_k$, are the numbers $1, 2, \ldots, p'$ in a certain order,

$$A = \tfrac{1}{8}(p^2 - 1) - B.$$

Upon substitution in (4), the resulting relation may be written

$$\mu p + M \equiv \tfrac{1}{24}p(p^2-1) + \tfrac{1}{8}(p^2-1) (\mathrm{mod}\, 2).$$
$$\therefore \mu \equiv M + \tfrac{1}{24}(p+3)(p^2-1) (\mathrm{mod}\, 2).$$

Now for $p = 4n+3$, $\frac{1}{24}(p+3)(p^2-1) = \frac{1}{3}(2n+3)(n+1)(4n+2) \equiv 0 (\mathrm{mod}\, 2)$. Hence the interesting conclusion, $\mu \equiv M (\mathrm{mod}\, 2)$. The number M may be expressed by means of circular functions on applying the relation

$$\left[\frac{m}{n}\right] = \frac{m}{n} - \tfrac{1}{2} + \frac{1}{2n}\sum_{k=1}^{n-1} \sin\frac{2km\pi}{n} \cot\frac{k\pi}{n},$$

where m and n are positive integers, n not a factor of m (Eisenstein, *Crelle*, XXVII, p. 281).

The residue of μ modulo 2 occurs in the investigation of the following question: If a is any primitive root of the prime $p=4n+3$, then the least absolute residues of $a, a^2, \ldots, a^{\frac{1}{2}(p-1)}$, modulo p, are the integers $1, 2, 3, \ldots, \frac{1}{2}(p-1)$, in some order. For no two of them can be equal in absolute value or we would have a congruence of the form

$$a^\alpha \equiv \pm a^\beta, \quad a^{\alpha-\beta} \equiv \pm 1 \pmod{p} \qquad [\beta < \alpha < \tfrac{1}{2}(p-1)]$$

which is impossible since $\alpha-\beta$ is less than $\frac{1}{2}(p-1)$. The residues are then distinct, and we get by multiplication, ν being the number of residues in question which are negative,

$$a^{\frac{1}{2}(p^2-1)} \equiv (-1)^\nu \frac{p-1}{2}! \pmod{p}.$$

But $a^{\frac{1}{2}(p^2-1)} \equiv (-1)^{n+1}$, $\frac{1}{2}(p-1)! \equiv (-1)^\mu \pmod{p}$. $\therefore \nu = \mu + n + 1 \pmod{2}$.

We might ask whether there is not a method for determining $\mu \pmod 2$, based on a theory of distribution of quadratic residues. But there appears to be but one general fact known regarding the residues less than $\frac{1}{2}p$, namely,

If p is a prime of the form $3 \pmod 4$, *then in the series* $1, 2, 3, \ldots, \frac{1}{2}(p-1)$, *there are more quadratic residues of p than non-residues.*

It may be mentioned here that no elementary proof of this theorem has as yet been given.*

If in the first congruence of the paper we put $p=4m+1$, then

$$\left(\frac{p-1}{2}!\right)^2 \equiv -1 \pmod{p}.$$

Since we can now set $p=a^2+b^2$, we may write

$$\frac{p-1}{2}! \equiv \pm i \pmod{a+bi}, \quad i^2 = -1.$$

The theorems concerning this two-valued unit residue are analogous to those which have been developed for the case $p=4n+3$.

For example, make $p=8k+5$, and calculate the least positive residues of (3), modulo p. If m is one of these residues then $p-m$ is also. For, if a^α is the term of the series corresponding to m, then there is one and only one positive integer β, less than $\frac{1}{2}(p-1)$, such that

$$\beta^2 \equiv p - a^2 \pmod{p},$$

as follows from the fact that p has the form $1 \pmod 4$ (Bachmann, l. c., p. 195).

*For what appears to be the only proof known, and which is not elementary, see Smith, *Works*, Vol. I, p. 275, 3rd foot-note.

Hence the residues may be put into two classes, $2k+1$ of them less than $4k+2$, and $2k+1$ of them greater than $4k+2$. Represent these by $r_1, r_2, \ldots, r_{2k+1}$, and $s_1, s_2, \ldots, s_{2k+1}$, respectively. Now find a positive integer N_γ such that

$$S_\gamma \equiv \pm iN_\gamma \pmod{a+bi}$$

where γ is arbitrary and $N_\gamma < \frac{1}{2}(p-1)$. As this involves the solution of a linear congruence in the domain $R(i)$ the unique determination of N_γ is always possible. Hence the residues of (3) can be expressed by

$$r_1, r_3, \ldots, r_{2k-1}, iN_1, iN_2, \ldots, iN_{2k+1} \ldots (5),$$

if we disregard order and sign. Now there cannot be a relation

$$r_\delta \equiv N_\epsilon \pmod{a+bi}.$$

For if f^2 and g^2 are two terms of the series (3) such that $f^2 \equiv r_\delta$, $g^2 \equiv \pm iN_\epsilon \pmod{a+bi}$, then

$$f^2 \equiv \pm ig^2, f^4 + g^4 \equiv 0 \pmod{a+bi}.$$

$\therefore f^4 + g^4 \equiv 0 \pmod{p}$. But this last relation is impossible since p has the form $5 \pmod 8$. Also it is obvious that we cannot have such relations as $r_n \equiv r_\nu$, $N_n \equiv N_\nu \pmod{a+bi}$, where $n = 1, 2, \ldots, 2k+1$, $\nu = 1, 2, \ldots, 2k+1$. Hence the series (5) is identical with $1, 2, \ldots, 4k+2$, in some order. Multiplication then gives, using (3) and (5),

$$[(4k+2)!]^2 \equiv (-1)^{\mu_1} i^{2k+1} (4k+2)! \pmod{a+bi}.$$

Dividing by $(4k+2)!$ which is prime to $a+bi$, and reducing,

$$(4k+2)! \equiv (-1)^{\mu_1+1} i \pmod{a+bi},$$

where μ_1 is the number of negative terms in (5) which are obtained when the N's are given their proper signs.

EXAMPLE. Let $p=13$. Then the numerically least residues of

$$1^2, 2^2, 3^2, 4^2, 5^2, 6^2,$$

are $1, 4, -4, 3, -1, -3$. But $-4 \equiv 6i$, $-3 = -2i$, $-1 \equiv -5i \pmod{2+3i}$. $\therefore 6! \equiv -i \equiv 5 \pmod{2+3i}$, $6! \equiv 5 \pmod{13}$.

To develop theorems corresponding to the above for the determination of the least residue of $\left(\frac{q-1}{2}\right)! \pmod q$, where q is a prime of the form $1 \pmod 8$, it would be necessary to make use of complex moduli defined by higher cyclotomic equations. But the treatment of the case $p \equiv 5 \pmod 8$ is sufficient to show the

character of the analysis one would encounter. It is curious that the ambiguities already discussed have an analogue in the following theorem:*

If P is a prime of the form $4c+1$, and $m^2+n^2=P$ (m and n integers), then either m or n has the form $\pm 1(\mathrm{mod}4)$. If $m\equiv\pm 1(\mathrm{mod}4)$, then

$$m\equiv\pm\frac{(2c)!}{2(c!)^2}(\mathrm{mod}P).$$

Now since $(2c)!\equiv -1(\mathrm{mod}P)$, we can write

$$(c!)^2\equiv\pm\frac{1}{2m}(\mathrm{mod}P)\ldots\ldots(7).$$

Hence the least residue of $(c!)^4(\mathrm{mod}P)$ can be determined uniquely. But in (7) the residue for $(c!)^2(\mathrm{mod}P)$ is two-valued, and we face a problem which is analogous to Dirichlet's.

ON THE EVALUATION OF CERTAIN DEFINITE INTEGRALS.

By PROFESSOR G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

1. In works on Calculus we find the following:

$$\int_0^\infty e^{-[x^2+(c^2/x^2)]}dx=\tfrac{1}{2}\sqrt{\pi}\,e^{-2c}.$$

We wish to find the value of

$$\int_0^\infty e^{-[fx^2+(g/x^2)]}\sin[hx^2+(k/x^2)]dx=A,$$

$$\int_0^\infty e^{-[fx^2+(g/x^2)]}\cos[hx^2+(k/x^2)]dx=B,$$

as well as of the still more general definite integral C of §6.

In view of the substitution $y=x\sqrt{m}$, we have

$$\int_0^\infty e^{-[mx^2+(n/x^2)]}dx=\frac{1}{\sqrt{m}}\int_0^\infty e^{-[y^2+(mn/y^2)]}dy=\frac{\sqrt{\pi}}{2\sqrt{m}}e^{-2\sqrt{(mn)}}.$$

Now $A=\dfrac{1}{2\sqrt{-1}}\displaystyle\int_0^\infty\{e^{-(f-h\sqrt{-1})x^2-(g-k\sqrt{-1})(1/x^2)}-e^{-(f+h\sqrt{-1})x^2-(g+k\sqrt{-1})(1/x^2)}\}dx.$

*This theorem is due to Gauss; see Smith, l. c., p. 268.

$$B=\tfrac{1}{2}\int_0^\infty [e^{-(f-h\sqrt{-1})x^2-(g-k\sqrt{-1})(1/x^2)}+e^{-(f+h\sqrt{-1})x^2-(g+k\sqrt{-1})(1/x^2)}]dx.$$

Let $\sqrt{m}=\sqrt{[f\pm h\sqrt{(-1)}]}=\sqrt{[\frac{1}{2}f+\frac{1}{2}\sqrt{(f^2+h^2)}]}\pm\sqrt{[\frac{1}{2}f-\frac{1}{2}\sqrt{(f^2+h^2)}]}=r\pm t,$

$$\sqrt{n}=\sqrt{[g\pm k\sqrt{(-1)}]}=\sqrt{[\tfrac{1}{2}g+\tfrac{1}{2}\sqrt{(g^2+k^2)}]}\pm\sqrt{[\tfrac{1}{2}g-\tfrac{1}{2}\sqrt{(g^2+k^2)}]}=u\pm v.$$

$$\therefore A=\frac{\sqrt{\pi}}{4\sqrt{(-1)}\,(r-t)}e^{-2(r-t)(u-v)}-\frac{\sqrt{\pi}}{4\sqrt{(-1)}\,(r+t)}e^{-2(r+t)(u+v)},$$

$$B=\frac{\sqrt{\pi}}{4(r-t)}e^{-2(r-t)(u-v)}+\frac{\sqrt{\pi}}{4(r+t)}e^{-2(r+t)(u+v)}.$$

$$A=\frac{\sqrt{\pi}}{4\sqrt{(-1)}\,(r^2-t^2)}e^{-2(ru+tv)}\{r[e^{2(rv+tu)}-e^{-2(rv+tu)}]+t[e^{2(rv+tu)}+e^{-2(rv+tu)}]\}$$

$$=-\frac{\sqrt{\pi}\sqrt{-1}}{2(r^2-t^2)}e^{-2(ru+tv)}\{r\sinh 2(rv+tu)+t\cosh 2(rv+tu)\}.$$

Similarly,

$$B=\frac{\sqrt{\pi}}{2(r^2-t^2)}e^{-2(ru+tv)}\{r\cosh 2(rv+tu)+t\sinh 2(rv+tu)\}.$$

$$\therefore \int_0^\infty e^{-|fx^2+(g/x^2)}\begin{matrix}\cos\\ \sin\end{matrix}\left[hx^2+\frac{k}{x^2}\right]dx$$

$$=\pm\frac{\sqrt{\pi}\sqrt{(\pm 1)}}{2(r^2-t^2)}e^{-2(ru+tv)}\left[r\begin{matrix}\cosh\\ \sinh\end{matrix}\Big(2(rv+tu)\Big)+t\begin{matrix}\sinh\\ \cosh\end{matrix}\Big(2(rv+tu)\Big)\right].$$

2. Let $f=1$, $g=0$, $h=0$, $k=b^2$; then $r=1$, $t=0$, $u=\frac{1}{2}b\sqrt{2}$, $v=\frac{1}{2}b\sqrt{-2}$, $r^2-t^2=1$, $ru+tv=\frac{1}{2}b\sqrt{2}$, $rv+tu=\frac{1}{2}b\sqrt{-2}$.

$$\therefore A=-\frac{\sqrt{\pi}\sqrt{(-1)}}{2}e^{-b\sqrt{2}}\sinh(b\sqrt{-2})=-\frac{\sqrt{\pi}}{2}e^{-b\sqrt{2}}\sqrt{(-1)}\sinh(b-\sqrt{2})$$

$$=-\frac{\sqrt{\pi}}{2}e^{-b\sqrt{2}}\sin(b\sqrt{-2}\sqrt{-1})=\tfrac{1}{2}\sqrt{(\pi)}e^{-b\sqrt{2}}\sin(b\sqrt{2})=\int_0^\infty e^{-x^2}\sin(b^2/x^2)dx.$$

Similarly, $B=\int_0^\infty e^{-x^2}\cos(b^2/x^2)dx=\frac{1}{2}\sqrt{(\pi)}e^{-b\sqrt{2}}\cos(b\sqrt{2})$.

3. Let $f=\cos\theta$, $g=a^2\cos\theta$, $h=\sin\theta$, $k=a^2\sin\theta$; then $r=\cos\frac{1}{2}\theta$, $t=\sqrt{(-1)}\times\sin\frac{1}{2}\theta$, $u=a\cos\frac{1}{2}\theta$, $v=a\sqrt{(-1)}\sin\frac{1}{2}\theta$, $r^2-t^2=1$, $ru+tv=a\cos\theta$, $rv+tu=a\sqrt{(-1)}\sin\theta$.

$$\therefore A=\frac{\sqrt{\pi}}{2}e^{-2a\cos\theta}\{-\sqrt{-1}\cos\tfrac{1}{2}\theta\sinh(2a\sqrt{-1}\sin\theta)$$

$$-\sqrt{-1}(\sqrt{-1}\sin\tfrac{1}{2}\theta)\cosh(2a\sqrt{-1}\sin\theta)\}$$

$$=\frac{\sqrt{\pi}}{2}e^{-2a\cos\theta}\{\cos\tfrac{1}{2}\theta\sin(2a\sin\theta)+\sin\tfrac{1}{2}\theta\cos(2a\sin\theta)\}$$

$$=\frac{\sqrt{\pi}}{2}e^{-2a\cos\theta}\sin(2a\sin\theta+\tfrac{1}{2}\theta)=\int_0^\infty e^{-[x^2+(a^2/x^2)]\cos\theta}\sin\{[x^2+(a^2/x^2)]\sin\theta\}dx.$$

Similarly,

$$B=\int_0^\infty e^{-[x^2+(a^2/x^2)]}\cos\{[x^2+(a^2/x^2)]\sin\theta\}dx=\frac{\sqrt{\pi}}{2}e^{-2a\cos\theta}\cos(2a\sin\theta+\tfrac{1}{2}\theta).$$

4. Let $f=1$, $g=b^2$, $h=1$, $k=b^2$; then $r=\frac{1}{2}\sqrt{(2\sqrt{2}+2)}$, $t=\frac{1}{2}\sqrt{(2-2\sqrt{2})}$, $u=\frac{1}{2}b\sqrt{(2\sqrt{2}+2)}$, $v=\frac{1}{2}b\sqrt{(2-2\sqrt{2})}$, $r^2-t^2=\sqrt{2}$, $ru+tv=b$, $rv+tu=b\sqrt{-1}$.

$$\therefore A=\int_0^\infty e^{-[x^2+(b^2/x^2)]}\sin[x^2+(b^2/x^2)]dx$$

$$=\frac{\sqrt{\pi}}{4\sqrt{2}}e^{-2b}\{-\sqrt{-1}\sqrt{(2+2\sqrt{2})}\sinh(2b\sqrt{-1})$$

$$-\sqrt{-1}\sqrt{(2-2\sqrt{2})}\cosh(2b\sqrt{-1})\}$$

$$=\frac{\sqrt{\pi}}{4}e^{-2b}\{\sqrt{(\sqrt{2}+1)}\sin(2b)-\sqrt{(\sqrt{2}-1)}\cos(2b)\},$$

$$B=\int_0^\infty e^{-[x^2+(b^2/x^2)]}\cos[x^2+(b^2/x^2)]dx$$

$$=\frac{\sqrt{\pi}}{4}e^{-2b}\{\sqrt{(\sqrt{2}+1)}\cos(2b)-\sqrt{(\sqrt{2}-1)}\sin(2b)\}.$$

5. Let $f=\cos2\theta$, $g=\frac{1}{2}a^2\sin2\theta$, $h=\sin2\theta$, $k=\frac{1}{2}a^2\cos2\theta$; then $r=\cos\theta$, $t=\sqrt{(-1)}\sin\theta$, $u=\frac{1}{2}a\sqrt{(1+\sin2\theta)}=\frac{1}{2}a(\cos\theta+\sin\theta)$, $v=\frac{1}{2}a\sqrt{(\sin2\theta-1)}=\frac{1}{2}a\sqrt{-1}(\cos\theta-\sin\theta)$, $r^2-t^2=1$, $ru+tv=\frac{1}{2}a$, $rv+tu=\frac{1}{2}a\sqrt{-1}$.

$$\therefore A=\int_0^\infty e^{-[x^2\cos2\theta+(a^2/2x^2)\sin2\theta]}\sin[x^2\sin2\theta+(a^2/2x^2)\cos2\theta]dx$$

$$=\tfrac{1}{2}\sqrt{(\pi)}e^{-a}\{-\sqrt{(-1)}\cos\theta\sinh(a\sqrt{-1})-\sqrt{(-1)}[\sqrt{(-1)}\sin\theta]\cosh(a\sqrt{-1})\}$$

$$=\tfrac{1}{2}\sqrt{(\pi)}e^{-a}\{\cos\theta\sin a+\sin\theta\cos a)=\tfrac{1}{2}\sqrt{(\pi)}e^{-a}\sin(\theta+a),$$

$$B=\int_0^\infty e^{-[x^2\cos2\theta+(a^2/2x^2)\sin2\theta]}\cos[x^2\sin2\theta+(a^2/2x^2)\cos2\theta]dx=\tfrac{1}{2}\sqrt{(\pi)}e^{-a}\cos(\theta+a).$$

6. Let $C=\int_0^\infty e^{-(fx^2+g/x^2)}\sin(hx^2+k/x^2)\cos(px^2+q/x^2)dx$

$$=\frac{1}{4\sqrt{-1}}\int_0^\infty e^{-(fx^2+g/x^2)}\left(e^{(hx^2+k/x^2)\sqrt{-1}}-e^{-(hx^2+k/x^2)\sqrt{-1}}\right)$$
$$\times\left(e^{(px^2+q/x^2)\sqrt{-1}}+e^{-(px^2+q/x^2)\sqrt{-1}}\right)dx.$$

$$\therefore C=\frac{1}{4\sqrt{-1}}\int_0^\infty\left[e^{-[f-(h+p)\sqrt{-1}]x^2-[g-(k-q)\sqrt{-1}]1/x^2}\right.$$
$$\left.-e^{-[f+(h+p)\sqrt{-1}]x^2-[g+(k+q)\sqrt{-1}]1/x^2}\right]dx$$

$$+\frac{1}{4\sqrt{-1}}\int_0^\infty\left[e^{-[f-(h-p)\sqrt{-1}]x^2-[g-(k-q)\sqrt{-1}]1/x^2}-e^{-[f+(h-p)\sqrt{-1}]x^2-[g+(k-q)\sqrt{-1}]1/x^2}\right]dx.$$

Let $\sqrt{\{f\pm(h+p)\sqrt{-1}\}}=\sqrt{\{\frac{1}{2}f+\frac{1}{2}\sqrt{[f^2+(h+p)^2]}\}}$
$$\pm\sqrt{\{\tfrac{1}{2}f-\tfrac{1}{2}\sqrt{[f^2+(h+p)^2]}\}}=r\pm s,$$
$\sqrt{\{g\pm(k+q)\sqrt{-1}\}}=\sqrt{\{\frac{1}{2}g+\frac{1}{2}\sqrt{[g^2+(k+q)^2]}\}}$
$$\pm\sqrt{\{\tfrac{1}{2}g-\tfrac{1}{2}\sqrt{[g^2+(k+q)^2]}\}}=t\pm u,$$
$\sqrt{\{f\pm(h-p)\sqrt{-1}\}}=\sqrt{\{\frac{1}{2}f+\frac{1}{2}\sqrt{[f^2+(h-p)^2]}\}}$
$$\pm\sqrt{\{\tfrac{1}{2}f-\tfrac{1}{2}\sqrt{[f^2+(h-p)^2]}\}}=v\pm w,$$
$\sqrt{\{g\pm(k-q)\sqrt{-1}\}}=\sqrt{\{\frac{1}{2}g+\frac{1}{2}\sqrt{[g^2+(k-q)^2]}\}}$
$$\pm\sqrt{\{\tfrac{1}{2}g-\tfrac{1}{2}\sqrt{[g^2+(k-q)^2]}\}}=y\pm z.$$

$$\therefore C=\frac{\sqrt{\pi}}{8\sqrt{-1}}\left(\frac{e^{-2(r-s)(t-u)}}{r-s}-\frac{e^{-2(r+s)(t+u)}}{r+s}+\frac{e^{-2(v-w)(y-z)}}{v-w}-\frac{e^{-2(v+w)(y+z)}}{v+w}\right)$$

$$=-\frac{\sqrt{\pi}\sqrt{-1}}{4(r^2-s^2)}e^{-2(rt+su)}[r\sinh 2(ru+ts)+s\cosh 2(ru+ts)]$$

$$-\frac{\sqrt{\pi}\sqrt{-1}}{4(v^2-w^2)}e^{-2(vy+wz)}[v\sinh 2(vz+wy)+w\cosh 2(vz+wy)].$$

7. Let $f=1$, $k=a^2$, $q=b^2$, $g=h=p=0$; then $r=v=1$, $s=w=0$,

$$t=\tfrac{1}{2}\sqrt{[2(a^2+b^2)]},\quad u=\frac{\sqrt{-1}}{2}\sqrt{[2(a^2+b^2)]},\quad y=\tfrac{1}{2}\sqrt{[2(a^2-b^2)]},$$

$$z=\frac{\sqrt{-1}}{2}\sqrt{[2(a^2-b^2)]},\quad r^2-s^2=v^2-w^2=0,\quad rt+su=\tfrac{1}{2}\sqrt{[2(a^2+b^2)]},$$

$$ru+ts=\frac{\sqrt{-1}}{2}\sqrt{[2(a^2+b^2)]},\quad vy+wz=\tfrac{1}{2}\sqrt{[2(a^2-b^2)]},$$

$$vz+wy=\frac{\sqrt{-1}}{2}\sqrt{[2(a^2-b^2)]}.$$

$$\therefore C=\int_0^\infty e^{-x^2}\sin(a^2/x^2)\cos(b^2/x^2)dx=\frac{\sqrt{\pi}}{4}\{e^{-\sqrt{[2(a^2+b^2)]}}\sin\sqrt{[2(a^2+b^2)]}$$

$$+e^{-\sqrt{[2(a^2-b^2)]}}\sin\sqrt{[2(a^2-b^2)]}\}.$$

If $b=0$, $C=\int_0^\infty e^{-x^2}\sin(a^2/x^2)dx=\frac{\sqrt{\pi}}{2}e^{-a\sqrt{2}}\sin(a\sqrt{2})$.

8. If we integrate both sides of the equation

$$\int_0^\infty e^{-mx}dx=\frac{1}{m}$$

$(n+1)$ times with respect to m, we get

$$(-1)^{n+1}\int_0^\infty \frac{e^{-mx}}{x^{n+1}}dx=\frac{m^n}{n!}\log m-\frac{m^n}{n!}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots\ldots+1/n).$$

From $\int_0^\infty\int_b^a e^{-mx}dxdm=\int_b^a\frac{dm}{m}$, we get $\int_0^\infty\frac{e^{-ax}-e^{-bx}}{x}dx=\log\left(\frac{b}{a}\right)$ .....(p).

From $-\int_0^\infty\int_{a+b}^{2a}\frac{e^{-mx}}{x}dxdm-\int_0^\infty\int_{a+b}^{2b}\frac{e^{-mx}}{x}dxdm=\int_{a+b}^{2a}\log m dm+\int_{a+b}^{2b}\log m dm$

we get $\int_0^\infty\left(\frac{e^{-ax}-e^{-bx}}{x}\right)^2dx=2a\log 2a+2b\log 2b-2(a+b)\log(a+b)$

$$=\log\left(\frac{(2a)^{2a}(2b)^{2b}}{(a+b)^{2(a+b)}}\right)\text{.....}(q).$$

Now $\int_0^\infty\frac{e^{-ax}-e^{-bx}}{x}\cos rx dx=\frac{1}{2}\int_0^\infty\frac{(e^{-ax}-e^{-bx})(e^{rx\sqrt{-1}}+e^{-rx\sqrt{-2}})}{x}dx$

$=\frac{1}{2}\int_0^\infty\left(e^{-x(a-r\sqrt{-1})}-e^{-x(b-r\sqrt{-1})}+e^{-x(a+r\sqrt{-1})}-e^{-x(b+r\sqrt{-1})}\right)x^{-1}dx$

$=\frac{1}{2}\log\left(\frac{b-r\sqrt{-1}}{a-r\sqrt{-1}}\right)+\frac{1}{2}\log\left(\frac{b+r\sqrt{-1}}{a+r\sqrt{-1}}\right)=\frac{1}{2}\log\left(\frac{b^2+r^2}{a^2+r^2}\right)$, by (p).

$$\int_0^\infty\frac{e^{-ax}-e^{-bx}}{x}\sin rx\,dx=\frac{1}{2\sqrt{-1}}\int_0^\infty\frac{(e^{-ax}-e^{-bx})(e^{rx\sqrt{-1}}-e^{-rx\sqrt{-1}})}{x}\Big)dx$$

$$=\frac{1}{2\sqrt{-1}}\int_0^\infty\left(e^{-x(a-r\sqrt{-1})}-e^{-x(b-r\sqrt{-1})}-e^{-x(a+r\sqrt{-1})}+e^{-x(b+r\sqrt{-1})}\right)x^{-1}dx$$

$$=\frac{1}{2\sqrt{-1}}\log\left(\frac{b-r\sqrt{-1}}{a-r\sqrt{-1}}\right)-\frac{1}{2\sqrt{-1}}\log\left(\frac{b+r\sqrt{-1}}{a+r\sqrt{-1}}\right)=\tan^{-1}\frac{r}{a}-\tan^{-1}\frac{r}{b}.$$

This last follows from the relations

$$\log(C \pm D\sqrt{-1}) = \tfrac{1}{2}\log(C^2 + D^2) \pm \sqrt{(-1)}\tan^{-1}(D/C).$$

$$\int_0^\infty \left(\frac{e^{-ax} - e^{-bx}}{x}\right)\cos^2 rx\, dx = \tfrac{1}{4}\int_0^\infty \left(\frac{(e^{-ax} - e^{-bx})\,(e^{rx\sqrt{-1}} + e^{-rx\sqrt{-1}})}{x}\right)^2 dx$$

$$= \tfrac{1}{4}\int_0^\infty x^{-2}(e^{-x(a-r\sqrt{-1})} - e^{-x(b-r\sqrt{-1})} + e^{-x(a+r\sqrt{-1})} - e^{-x(b+r\sqrt{-1})})^2 dx$$

$$= \tfrac{1}{4}\int_0^\infty x^{-2}(e^{-x(a-r\sqrt{-1})} - e^{-x(b-r\sqrt{-1})})^2 dx + \tfrac{1}{2}\int_0^\infty x^{-2}(e^{-ax} - e^{-bx})^2 dx$$

$$+ \tfrac{1}{4}\int_0^\infty x^{-2}(e^{-x(a+r\sqrt{-1})} - e^{-x(b+r\sqrt{-1})})^2 dx$$

$$= \tfrac{1}{2}\log\left[\frac{(2a)^{2a}(2b)^{2b}}{(a+b)^{2(a+b)}}\right] + \tfrac{1}{2}(a - r\sqrt{-1})\log[2(a - r\sqrt{-1})]$$

$$+ \tfrac{1}{2}(b - r\sqrt{-1})\log[2(b - r\sqrt{-1})] - \tfrac{1}{2}(a + b - 2r\sqrt{-1})\log(a + b - 2r\sqrt{-1})$$

$$+ \tfrac{1}{2}(a + r\sqrt{-1})\log[2(a + r\sqrt{-1}) + \tfrac{1}{2}(b + r\sqrt{-1})\log[2(b + r\sqrt{-1})]$$

$$- \tfrac{1}{2}(a + b + 2r\sqrt{-1})\log(a + b + 2r\sqrt{-1}), \text{ from } (q),$$

$$= \tfrac{1}{2}\log\left[\frac{(2a)^{2a}(2b)^{2b}}{(a+b)^{2(a+b)}}\right] + \tfrac{1}{2}a\log(4a^2 + 4r^2) + \tfrac{1}{2}b\log(4b^2 + 4r^2)$$

$$- \tfrac{1}{2}(a + b)\log[(a + b^2) + 4r^2]$$

$$= \tfrac{1}{2}\log\left[\frac{(2a)^{2a}(2b)^{2b}(4a^2 + 4r^2)^a(4b^2 + 4r^2)^b}{(a+b)^{2(a+b)}[(a+b)^2 + 4r^2]^{(a+b)}}\right].$$

Since $\sin^2 rx = 1 - \cos^2 rx$, we get

$$\int_0^\infty \left[\frac{e^{-ax} - e^{-bx}}{x}\right]^2 \sin^2 rx\, dx = \tfrac{1}{2}\log\left[\frac{(2a)^{2a}(2b)^{2b}[(a+b)^2 + 4r^2]^{(a+b)}}{(a+b)^{2(a+b)}\,(4a^2 + 4r^2)^a(4b^2 + 4r^2)^b}\right].$$

From $\int_0^\infty \int_{2a+b}^{3a} \frac{e^{-mx}}{x^2} dx\, dm + 2\int_0^\infty \int_{2a+b}^{a+2b} \frac{e^{-mx}}{x^2} dx\, dm + \int_0^\infty \int_{3b}^{a+2b} \frac{e^{-mx}}{x^2} dx\, dm$

$$= \int_{2a+b}^{3a} (m\log m - m)dm + 2\int_{2a+b}^{a+2b} (m\log m - m)dm + \int_{3b}^{a+2b} (m\log m - m)dm, \text{ we get}$$

$$\int_0^\infty \left[\frac{e^{-ax} - e^{-bx}}{x}\right]^3 dx = -\tfrac{1}{2}(3a)^2\log(3a) + \tfrac{3}{2}(2a + b)^2\log(2a + b)$$

$$-\tfrac{3}{2}(a+2b)^2\log(a+2b)+\tfrac{1}{2}(3b)^2\log(3b)=\tfrac{1}{2}\log\left[\frac{(2a+b)^{3(2a+b)^2}(3b)^{(3b)^2}}{(3a)^{(3a)^2}(a+2b)^{3(a+2b)^2}}\right].$$

$$-\int_0^\infty\int_{3a+b}^{4a}\frac{e^{-mx}}{x^3}dx\,dm\ -3\int_0^\infty\int_{3a+b}^{2a+2b}\frac{e^{-mx}}{x^3}dx\,dm$$

$$-3\int_0^\infty\int_{a+3b}^{2a+2b}\frac{e^{-mx}}{x^3}dx\,dm-\int_0^\infty\int_{a+3b}^{4b}\frac{e^{-mx}}{x^3}dx\,dm$$

$$=\int_{3a+b}^{4a}(\tfrac{1}{2}m^2\log m-\tfrac{3}{4}m^2)dm+3\int_{3a+b}^{2a+2b}(\tfrac{1}{2}m^2\log m-\tfrac{3}{4}m^2)dm$$

$$+3\int_{a+3b}^{2a+2b}(\tfrac{1}{2}m^2\log m-\tfrac{3}{4}m^2)dm+\int_{a+3b}^{4b}(\tfrac{1}{2}m^2\log m-\tfrac{3}{4}m^2)dm,\text{ we get}$$

$$\int_0^\infty\left[\frac{e^{-ax}-e^{-bx}}{x}\right]^4dx=\tfrac{1}{6}\log\left[\frac{(4a)^{(4a)^3}(2a+2b)^{6(2a+2b)^3}(4b)^{(4b)^3}}{(3a+b)^{4(3a+b)^3}(a+3b)^{4(a+3b)^3}}\right].$$

In general, $\int_0^\infty\left[\frac{e^{-ax}-e^{-bx}}{x}\right]^n dx$ equals

when n is even, $\frac{1}{(n-1)!}\times$

$$\log\left[\frac{[na]^{a_0}[(n-2)a+2b]^{a_2}\ldots\ldots[2a+(n-2)b]^{a_{n-2}}[nb]^{a_n}}{[(n-1)a+b]^{a_1}[(n-3)a+3b]^{a_3}\ldots\ldots[3a+(n-3)b]^{a_{n-3}}[a+(n-1)b]^{a_{n-1}}}\right];$$

and when n is odd, $\frac{1}{(n-1)!}\times$

$$\log\left[\frac{[(n-1)a+b]^{a_1}[(n-3)a+3b]^{a_3}\ldots\ldots[2a+(n-2)b]^{a_{n-2}}[nb]^{a_n}}{[na]^{a_0}[(n-2)a+2b]^{a_2}\ldots\ldots[3a+(n-3)]^{a_{n-3}}[a+(n-1)b]^{a_{n-1}}}\right],$$

where $a_0=(na)^{n-1}$, $a_1=n[(n-1)a+b]^{n-1}$,

$$a_2=\frac{n(n-1)}{2!}[(n-2)a+2b]^{n-1},\quad a_3=\frac{n(n-1)(n-2)}{3!}[(n-3)a+3b]^{n-1},$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$a_i=\frac{n(n-1)(n-2)(n-3)\ldots\ldots(n-i+1)}{i!}[(n-i)a+ib]^{n-1}.$$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Problems 189 and 190 were also solved by L. E. Newcomb, Los Gatos, California.

190. Proposed by A. H. HOLMES, Brunswick, Maine.

Find the general term of the series 2, 3, 7, 46, 2112, etc.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and GRACE M. BAREIS, Miss Roney's School, Bala, Pa.

Denoting the $n+1$st term by T_{n+1}, we have the recursion formula $T_{n+1}=T_n^2-n$.

191. Proposed by NELSON L. RORAY, Bridgeton, N. J.

Find a number such that if it be multiplied by 2, 3, 4, 5, and 6, the cyclical order of its digits will not be changed.

Remark by L. E. DICKSON, Ph. D., The University of Chicago.

The answer is the period 142857, of the fraction $\frac{1}{7}$. Solved in the MONTHLY of 1895, p. 13, Algebra problem number 32.

GEOMETRY.

Problem 212 was also solved by F. D. Posey, A. B., San Mateo, California.
Problem 213 was also solved by L. E. Newcomb, Los Gatos, California, and F. D. Posey, A. B., San Mateo, California.

215. Proposed by M. J. NEWELL, A. M., Evanston High School, Evanston, Ill.

No satisfactory solution has been received.

216. Proposed by JOHN J. QUINN, Warren High School, Warren, Pa.

Find, by plane geometry, the sides of a right triangle if the hypotenuce is 35, and the side of the inscribed squre is 12.

Solution by L. E. DICKSON, Ph. D., The University of Chicago.

"The inscribed square" to a right triangle is open to two interpretations: (1) the square has a side lying along the hypotenuse; (2) the square has two sides lying along the legs of the right triangle. For (1), the present problem is impossible (see problem 221). For the interpretation (2), denote the legs by a and b, $a \gtrless b$, whence $a^2+b^2=35^2$. By similar triangles, $a-12:12=a:b$, whence $ab=12(a+b)$. Adding and subtracting twice the latter from $a^2+b^2=1225$, we get

$$(a+b)^2-24(a+b)=1225, \quad (a-b)^2=1225-24(a+b).$$

Hence $a+b=49$, $a-b=7$, $a=28$, $b=21$. The geometric verification is evident.

Also solved by B. F. Finkel, A.M., M.Sc., 204 St. Marks Square, Philadelphia, Pa.; J. K. Ellwood, Principal Colfax Sub-District School, Pittsburg, Pa.; A. H. Holmes, Brunswick, Me.; and G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va.; L. E. Newcomb, Los Gatos, California; G. I. Hopkins, Warren High School, Warren, Pa.; J. Scheffer, Hagerstown, Md.

217. Proposed by G. W. DRAKE, Fayetteville, Ark.

If one of the principal axes of a cone which stands on a given base be always parallel to a given right line, the locus of the vertex is an equilateral hyperbola or a right line according as the base is a central conic or a parabola. [Exercise 40, page 94, C. Smith's *Solid Geometry*].

Solution by WILLIAM HOOVER, Ph. D., Professor of Mathematics in the Ohio State University, Athens, O.

Let the given central conic be

$$x^2/a+y^2/b=1,\ z=0 \quad \ldots\ldots(1),$$

and regarded as the central conicoid

$$x^2/a+y^2/b+z^2/c=1 \quad \ldots\ldots(2),$$

in which the semi-axis c vanishes.

This solution will employ the principle that the principal axes of a cone, which envelope a given conicoid, are normals to the three confocals which pass through the vertex. The confocals to (2) are given by

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{c+\lambda}=1 \quad \ldots\ldots(3),$$

and so the confocals to (1), by

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{\lambda}=1 \quad \ldots\ldots(4).$$

If (ξ, η, ζ) be the vertex of the cone, a normal to one of the system (4) is

$$\frac{[a+\lambda][x-\xi]}{\xi}+\frac{[b+\lambda][y-\eta]}{\eta}+\frac{[c+\lambda][z-\zeta]}{\zeta}=1 \quad \ldots\ldots(5),$$

and if l, m, n be the direction-cosines of the given line, and ρ an undetermined number, we should have, for the normal to (4) at (η, ξ, ζ),

$$l=\frac{\rho\xi}{a+\lambda},\quad m=\frac{\rho\eta}{b+\lambda},\quad n=\frac{\rho\zeta}{\lambda} \quad \ldots\ldots(6),$$

$$\text{giving } a+\lambda=\frac{\rho\xi}{l},\quad b+\lambda=\frac{\rho\eta}{m},\quad \lambda=\frac{\rho\zeta}{n} \quad \ldots\ldots(7),$$

$$\text{and } l\xi+m\eta+n\zeta=\rho\left(\frac{\xi^2}{a+\lambda}+\frac{\eta^2}{b+\lambda}+\frac{\zeta^2}{\lambda}\right)=\rho \quad \ldots\ldots(8),$$

since (ξ, η, s) is on (4). From the first two of (7),

$$a-b=\rho[\xi/l-\eta/m] \quad \text{.....(9),}$$

and from the first and third of (7), $a=\rho[\xi/l-s/n]$ .....(10).

(8) and (9) give $a-b=[l\xi+m\eta+ns][\xi/l-\eta/m]$ .....(11),

and (9) and (10) give $[a-b][\xi/l-s/n]=a[\xi/l-\eta/m]$ .....(12),

showing that the locus of the vertex, when the base is a central conic (1), is the intersection of the hyperboloid (11) by the plane (12), or an equilateral hyperbola.

For the second part, let the base be an indefinitely thin elliptic or hyperbolic paraboloid given by

$$x^2/a+y^2/b=2z \quad \text{.....(13),}$$

and the paraboloids confocal to (13), by

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}=2z+\lambda \quad \text{.....(14).}$$

The normals to (14) through (ξ, η, s) are given by

$$\frac{[a+\lambda][x-\xi]}{\xi}+\frac{[b+\lambda][y-\eta]}{\eta}=-\frac{z-s}{-1} \quad \text{.....(15),}$$

and we have again

$$l=\frac{\rho\xi}{a+\lambda}, \quad m=\frac{\rho\eta}{b+\lambda}, \quad n=-\rho \quad \text{.....(16),}$$

giving $a+\lambda=\rho\xi/l, \quad b+\lambda=\rho\eta/m$ .....(17).

These give $l\xi+m\eta+2ns=\rho\left(\frac{\xi^2}{a+\lambda}+\frac{\eta^2}{b+\lambda}-2s\right)=\rho\lambda=-n\lambda$ .....(18).

This gives $-\lambda=\frac{l\xi+m\eta+2ns}{n}$ .....(19).

Adding (19) to the first of (17) and substituting from (16),

$$[l^2-n^2]\xi+lm\eta+2lns-aln=0 \quad \text{.....(20).}$$

Eliminating λ and ρ from (16) and (17),

$$a-b=-n[\xi/l-\eta/m] \quad \text{.....(21).}$$

Therefore in this case the locus of the vertex is a straight line, the intersection of the planes (20) and (21).

218. Proposed by O. W. ANTHONY, DeWitt Clinton High School, New York City.

From a given triangle cut off an area equivalent to a given square by a line passing through a given point without the triangle.

I. Solution by F. D. POSEY, A. B., Instructor in Mathematics, St. Matthews School, San Mateo, Cal.

Let ABC be the given triangle and P the given point on the opposite side AC say from B. Let a be the side of the given square and suppose the part cut off to be a triangle.

Draw PE perpendicular to AC and PD parallel to AB cutting AC in D. Consider the problem solved and the line PQR cutting AC and AB respectively in Q and R, to be the required line making $AQR=a^2$. Let $AQ\equiv x$, $QD\equiv y$, $PE\equiv m$, $AD\equiv n$. Since AQR and PQD are similar,

$$\frac{x^2}{y^2}=\frac{AQR}{PQD}=\frac{2a^2}{my}, \quad \frac{x^2}{y^2}=\frac{2a^2}{m}.$$

Since $y=n-x$, $\frac{x^2}{n-x}=\frac{2a^2}{m}$ or $x^2+\frac{2a^2}{m}x+\frac{2a^2n}{m}$.

Completing the square, $x^2+\frac{2a^2}{m}x+\frac{a^4}{m^2}=\frac{a^2}{m}\left(2n+\frac{a^2}{m}\right)$.

Construct a third proportional p to m, a. $\therefore p=a^2/m$.

$\therefore x^2+2px+p^2=p[2n+p]$.

Construct $r=2n+p$ and s a mean proportional between p and r.

$\therefore x^2+2px+p^2=s^2$, $x+p=s$, $x=s-p$.

By constructing $AQ=s-p$, AQR may be shown to have the required area. If the part cut off is not a triangle it may be reduced to the preceding case by cutting off a triangle equivalent to the difference of the given triangle and the given square.

II. Solution by G. W. GREENWOOD, A. M. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

Lines cutting off a triangle of constant area from two intersecting lines envelope an hyperbola having the given lines as asymptotes.*

Construct hyperbolas for each pair of sides, and draw tangent to them through the given point.

If we are restricted to compass and ruler, we can construct the foci and the auxiliary circle, and by this means construct the tangents through the points.

Other solutions, which arrived too late for insertion, will be printed next month.

*Take the two lines as the axes of abscissae, the angle between them being w and the given area A. The area of the triangle formed by the axes and the line $x/a+y/b=1$ is $\frac{1}{2}ab\sin w=A$, whence the equation of the line becomes $(ya^2-2A)\sin w-a+x=0$. Differentiating with respect to a and eliminating, the equation of the envelop is found to be the hyperbola $xy=(1/2\sin w)(2A^2-A)$. Ed.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ABC be the given triangle, AB base, C the vertex. Let P be the given point without, $PA=b$, $\angle PAC=\theta$, $\angle BAC=A$, area of square$=m^2$, area of triangle$=\triangle$, $\triangle - m^2 = n$. Let the line from P cutting off the area m^2 meet AC in D, AB in E; $AD=y$, $AE=x$.

Then $\frac{1}{2}xy\sin A = m^2$ ..... (1).

$\frac{1}{2}by\sin\theta = \text{area}PAD$ ..... (2).

$\frac{1}{2}bx\sin[\theta+A] = \text{area}PAE$ ..... (3).

$\therefore \frac{1}{2}bx\sin[\theta+A] - \frac{1}{2}by\sin\theta = m^2$ ..... (4).

The value of y from (1) in (4) gives

$$x^2 - \frac{2m^2x}{b\sin[\theta+A]} = \frac{2m^2\sin\theta}{\sin A\sin[\theta+A]}.$$

$$\therefore x = \frac{m^2}{b\sin[\theta+A]} \pm \sqrt{\frac{2m^2\sin\theta}{\sin A\sin[\theta+A]} + \frac{m^4}{b^2\sin^2[\theta+A)}},$$

$$y = \frac{\sin[\theta+A]}{\sin\theta}\left[\pm\sqrt{\frac{2m^2\sin\theta}{\sin A\sin[\theta+A]} + \frac{m^4}{b^2\sin^2[\theta+A]}} - \frac{m^2}{b\sin[\theta+A]}\right].$$

If m^2 is to be the quadrilateral part of the triangle, substitute n for m^2 in the above values for x, y.

Two geometrical constructions which arrived too late for insertion will be printed next month.

219. On account of the pedagogical importance of this problem, the solutions are withheld until next month in order that they may all be printed in the same number.

CALCULUS.

Problem 173 was also solved by L. E. Newcomb, Los Gatos, California.

174. Proposed by B. F. FINKEL, A. M., 204 St. Marks Square, Philadelphia, Pa.

Integrate $\int_0^\infty \frac{\sinh px}{\sinh qx}\cos rx\,dx$, if $p^2 < q^2$.

I. Solution by WILLIAM HOOVER, Ph. D., Professor of Mathematics, Ohio State University, Athens, O.

$$I = \int_0^\infty \frac{\sinh px}{\sinh qx}\cos rx\,dx = \int_0^\infty \frac{e^{px}-e^{-px}}{e^{qx}-e^{-qx}}\cos rx\,dx \quad \text{..... (1).}$$

In (1), put $qx=\pi z$, or $x=[\pi/q]z$; then (1) is

$$I = \frac{\pi}{q}\int_0^\infty \frac{e^{az}-e^{-az}}{e^{\pi z}-e^{-\pi z}}\cos mz\,dz \quad \text{..... (2),}$$

in which $a=p\pi/q$, $m=r\pi/q$, and the required integral

$$I=\frac{\pi}{q}\cdot\frac{\sin a}{e^{m}+2\cos a+e^{-m}} \qquad \text{(3)},$$

as given by Williamson's *Integral Calculus*, Appleton Edition of 1884, p. 143, Ex. 8, and by Carr's *Synopsis of Mathematics*, Edition of 1886, Equation 2594, p. 386. I am indebted to the latter work for the outline of the integral in equation (2). If we make, in $I'=\int_0^\infty \frac{e^{ax}-e^{-ax}}{e^{\pi x}-e^{-\pi x}}\sin mx\,dx$, the indicated division,

$$I'=\int_0^\infty [e^{ax}+e^{-ax}]\sin mx[e^{-\pi x}+e^{-3\pi x}+e^{-5\pi x}+\text{etc.}]dx,$$

in which, if we remove brackets, there occur terms of the type $\int_0^\infty e^{-a'x}\sin bx\,dx$, whose integral is, $\frac{b}{a'^2+b^2}$, Williamson, Ex. 19, p. 117, and

$$I'=\sum\frac{m}{\{[2n-1]\pi-a'\}^2+m^2}+\sum\frac{m}{\{[2n-1]\pi+a'\}^2+m^2}.$$

We have, by Casey's *Treatise on Plane Trigonometry*, Ed. 1888, Ex. 33, p. 231,

$$\frac{\cos x+\cos a'}{1+\cos a'}=\left[1-\frac{x^2}{[\pi-a']^2}\right]\left[1-\frac{x^2}{[\pi+a']^2}\right]\left[1-\frac{x^2}{[3\pi-a']^2}\right]$$

$$\times\left[1-\frac{x^2}{[3\pi+a']^2}\right]\left[1-\frac{x^2}{]5\pi-a']^2}\right]\cdots\cdots$$

In this put $x=ix$, and then taking the derivative of the logarithm of both members, noting that $\cos ix=\frac{1}{2}[e^{-x}+e^{x}]$, we have

$$\tfrac{1}{2}\frac{e^{x}-e^{-x}}{e^{x}+2\cos a'+e^{-x}}=\frac{x}{(\pi-a')^2+x^2}+\frac{x}{(\pi-a')^2+x^2}$$

$$+\frac{x}{(3\pi-a')^2+x^2}+\frac{x}{(3\pi+a')^2+x^2}+\text{etc.}$$

If in this we put $x=m$, and in the above summation for I' we make $n=1, 2,$ etc., we have, finally,

$$I'=\int_0^\infty \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}\sin mx\,dx=\tfrac{1}{2}\frac{e^{m}-e^{-m}}{e^{m}+2\cos a+e^{-m}}.$$

If in this we put $m=i\theta$, noting that $\sin(i\theta x)=\frac{1}{2}(e^{\theta x}-e^{-\theta x})$,

$$\int_0^\infty \frac{(e^{ax}+e^{-ax})(e^{\theta x}-e^{m\theta x})}{e^{\pi x}-e^{-\pi x}}dx=\frac{\sin\theta}{\cos a+\cos\theta}.$$

In this, change a into im, θ into a, and we have

$$\int_0^\infty \frac{e^{ax}-e^{-ax}}{e^{\pi x}-e^{-\pi x}}\cos mx\,dx=\frac{\sin a}{e^m+2\cos a+e^{-m}}.$$

Restoring the values of a and m, (3) becomes

$$I=\frac{\pi}{q}\frac{\sin p\pi/q}{e^{r\pi/q}+2\cos p\pi/q+e^{-r\pi/q}}.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $x=\pi y/q$, $a=\pi p/q$, $b\sqrt{-1}=\pi r/q$.

$$A=\int_0^\infty \frac{\sinh px\cos rx}{\sinh qx}dx=\frac{\pi}{q}\int_0^\infty \frac{\sinh ay\cos b\sqrt{(-1)}y}{\sinh \pi y}$$

$$=\frac{\pi}{2q}\int_0^\infty \frac{(e^{ay}-e^{-ay})(e^{by}+e^{-by})}{e^{\pi y}-e^{-\pi y}}dy$$

$$=\frac{\pi}{2q}\int_0^\infty \left[\frac{e^{(a+b)y}-e^{-(a+b)y}}{e^{\pi y}-e^{-\pi y}}+\frac{e^{(a-b)y}-e^{-(a-b)y}}{e^{\pi y}-e^{-\pi y}}\right]dy$$

$$=\frac{\pi}{4q}\tan\tfrac{1}{2}(a+b)+\frac{\pi}{4q}\tan\tfrac{1}{2}(a-b).$$

(Williamson's *Integral Calculus*, p. 142).

$$\tan\tfrac{1}{2}(a+b)+\tan\tfrac{1}{2}(a-b)=\frac{2\sin a}{\cos a+\cos b}.$$

$$\therefore A=\frac{\pi}{2q}\cdot\frac{\sin a}{\cos a+\cos b}=\frac{\pi}{2q}\cdot\frac{\sin(\pi p/q)}{\cos(\pi p/q)+\cosh(\pi r/q)}.$$

175. Proposed by M. E. GRABER, A. B., Instructor in Mathematics and Physics in Heidelberg University, Tiffin, Ohio.

Find the volume of the cono-cuneus determined by $z^2+a^2y^2/x^2=c^2$, which is contained between the planes $x=0$ and $x=a$. Ans. $\frac{1}{2}\pi c^2 a$. [Todhunter's *Integral Calculus*, p. 189, No. 28].

I. Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

The section by a plane parallel to the yz plane is an ellipse whose semi-axes are c and cx/a, and whose area is therefore c^2x/a. Hence the required volume is $\frac{\pi c^2}{a}\int_0^a x\,dx$, or $\frac{1}{2}\pi a c^2$.

II. Solution by B. F. FINKEL, A. M., M. Sc., 204 St. Marks Square, Philadelphia, Pa.

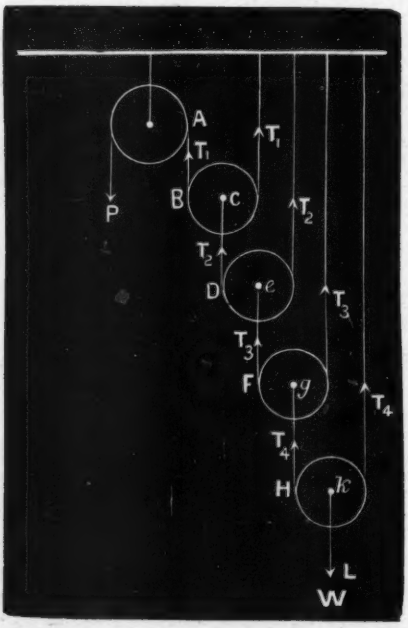
The equation may be written in the form $y=\pm\frac{x}{a}\sqrt{(c^2-z^2)}$. If $z=0$, $y=\pm\frac{c}{a}x$, the equation of two lines in the xy-plane passing through the origin; if $y=0$, $z=\pm c$, the equation of two lines in the xz-plane parallel to the x-axis. It is thus seen that the given surface intersects the xz-plane in two parallel straight lines, and the xy-plane in two straight lines passing through the origin. If $x=a$, $y=c^2-z^2$, the equation of a circle. The intersection of the surface with a plane parallel with the yz-plane at a distance $x=k<a$, is an ellipse. If we pass a plane parallel to the xy-plane at a distance $z<c$, the intersection of this plane with the surface and the plane $x=a$, is a triangle whose altitude is a, and base $2y=2a\sqrt{(c^2-z^2)}$. Hence we have for the required volume,

$$V=2\int_0^c 2a\sqrt{(c^2-z^2)}\,dz=4a\left[\frac{z}{2}\sqrt{(c^2-z^2)}+\frac{c^2}{2}\sin^{-1}\frac{z}{c}\right]_0^c=\tfrac{1}{2}\pi ac^2.$$

Also solved by W. W. Landis, Dickinson College, Carlisle, Pa.; E. L. Sherwood, Shady Side Academy, Pittsburg, Pa.; William Hoover, Athens, O.; G. B. M. Zerr, Parsons, W. Va.; L. C. Walker, Golden, Col.; J. Scheffer, Hagerstown, Md.; and the Proposer.

MECHANICS.

164. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.



P balances W on a system of n movable pulleys of equal weight, each hanging by a separate string. If P is moved find the maximum acceleration of W.

Solution by G B M ZERR, A M, Ph D, Parsons, W Va

Let T_1, T_2, T_3, ..., T_n be the tensions on AB, CD, EF, ..., GH; x_1, x_2, x_3, ..., x_n the lengths AB, CD, EF, ..., GH. Then dx_1/dt, dx_2/dt, dx_3/dt, ..., dx_n/dt represent the velocities.

Let Q be the additional weight which moves the system, m the weight of each pulley.

Now $P=T_2$, $2T_1=T_2+m$, $2T_2=T_3+m$, ..., $2T_n=m+W$.

$\therefore 2P=2T_1=T_2+m$, $4P=2T_2+2m=T_3+3m$, $2^3P=2T_3+6m=T_4+(2^3-1)m$, ..., $2^{n-1}P=T_n+(2^{n-1}-1)m$, $2^nP=(2^n-1)m+W$.

The equations of motion are

$$\left(\frac{P+Q}{g}\right)\cdot\frac{d^2x_1}{dt^2}=P+Q-T_1,$$

$$\frac{m}{g}\cdot\frac{d^2x_1}{dt^2}=2T_1-m-T_2,$$

$$\frac{m}{g}\cdot\frac{d^2x_2}{dt^2}=2T_2-m-T_3,\quad \frac{m}{g}\cdot\frac{d^2x_3}{dt^2}=2T_3-m-T_4,$$

....

$$\frac{m}{g}\cdot\frac{d^2x_{n-1}}{dt^2}=2T_{n-1}-m-T_n,\quad \left(\frac{m+W}{g}\right)\cdot\frac{d^2x_n}{dt^2}=2T_n-m-W.$$

Eliminating T_1 we get

$$\left(\frac{2(P+Q)}{g}+\frac{m}{2g}\right)\frac{d^2x_1}{dt^2}=2(P+Q)-m-T_2.$$

Now $dx_1/dt=2dx_2/dt=2^2dx_3/dt=\ldots\ldots=2^{n-1}dx_n/dt.$

$\therefore d^2x_1/dt^2=2d^2x_2/dt^2=2^2d^2x_3/dt^2=\ldots\ldots=2^{n-1}d^2x_n/dt^2.$

Eliminating T_2 we get

$$\left(\frac{2^2(P+Q)}{g}+\frac{m}{g}+\frac{m}{2^2g}\right)\cdot\frac{d^2x_1}{dt^2}=2^2(P+Q)-2m-m-T_3.$$

Eliminating T_3 we get

$$\left(\frac{2^3(P+Q)}{g}+\frac{2m}{g}+\frac{m}{2g}+\frac{m}{2^3g}\right)\cdot\frac{d^2x_1}{dt^2}=2^3(P+Q)-2^2-2m-m-T_4.$$

Hence eliminating all the tensions we get

$$\frac{2^n(P+P)}{g}+\frac{m}{g}\left(2^{n-2}+2^{n-4}+2^{n-6}+\ldots\ldots+\frac{1}{2^{n-4}}+\frac{1}{2^{n-2}}+\frac{1}{2^n}\right)\cdot\frac{d^2x_1}{dt^2}$$

$$=2^n(P+Q)-m(2^{n-1}+2^{n-2}+2^{n-3}+\ldots\ldots+2^2+2+1)-W$$

$$=2^n(P+Q)-m(2^n-1)-W.$$

$$\therefore\ \frac{1}{2^ng}\cdot\frac{d^2x_1}{dt^2}\left(2^{2n}(P+Q)+m(2^{2n-2}+2^{2n-4}+2^{2n-6}+\ldots\ldots+2^4+2^2+1)\right)$$

$$=2^{2n}(P+Q)-m(2^n-1)-W.$$

$$\therefore\ \frac{1}{2^n}\cdot\frac{d^2x_1}{dt^2}=\frac{[2^n(P+Q)-m(2^n-1)-W\,]g}{2^{2n}(P+Q)+\frac{1}{3}m(2^{2n}-1)}=\text{acceleration of } W=A.$$

Substituting the value of P we get

$$A=\frac{3.2^nQg}{3.2^{2n}Q+3.2^nW+3.2^n(2^n-1)m+(2^{2n}-1)m}$$

$$=\frac{3.2^nQg}{3.2^{2n}Q+3.2^nW+(4.2^{2n}-3.2^n-1)m}.$$

When Q is infinite A is a maximum$=g/2^n$.

165. Proposed by O. W. ANTHONY, DeWitt Clinton High School, New York City.

Find the approximate form of a tower of circular cross section 1000 feet high and having a radius of lower base 20 feet, and so constructed that all the parts of the structure shall be subject to the same stress, due to the weight of the part of the tower above.

I. Solution by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

Denote the radius of the section by r, and the height by h. The decrease in area of the sections parallel to the base is proportional to the decrease in the weight.

$\therefore -rdr=kr^2dh$. $\therefore \log(20/r)=kh$, since $r=20$ when $h=0$.

If the radius at the top be a, $\log(20/a)=1000k$.

$\therefore 1000\log(20/r)=h\log(20/a)$.

Uniform stress can only be secured by having a weight at the summit. The tower cannot come to a point.

II. Solution by S. A. COREY, Hiteman, Iowa.

Let x=vertical downward distance from the top of the tower. In order that all the parts of the structure may be subject to the same stress, the area of the circular cross section at any point must be proportional to the weight of the tower above. This condition will be approximately satisfied by assuming the area of the circular cross section to be e^{mx}, as then the volume of the part above will be $\int_0^x e^{mx}dx$, or e^{mx}/m, and the weight is directly proportional to the volume.

From $e^{mx}=\pi 20^2$=area of base, when $x=1000$, follows $m=.007136$. If the weight of $1/m$ cubic feet (140 cubic feet) of the same material used in the tower be evenly distributed on the square foot of summit the given conditions are all approximately satisfied.

Also solved by G. B. M. Zerr, Parsons, W. Va.

MISCELLANEOUS.

143. Proposed by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

If $A+B+C=180°$. show that $1-\cos^2A-\cos^2B-\cos^2C-2\cos A\cos B\cos C=0$.

Solution by MARY STEAGALL, Student at The University of Chicago.

Take $C=180°-(A+B)$. Then $\cos C=-\cos(A+B)$ and $\cos^2C=\cos^2(A+B)$. Introducing these values for $\cos C$ and $\cos^2C$ we have:

$$1-\cos^2A-\cos^2B-(\cos A\cos B-\sin A\sin B)^2+2\cos A\cos B(\cos A\cos B-\sin A\sin B)$$

Performing the operations indicated and putting $1-\cos^2A$ for $\sin^2A$, and $1-\cos^2B$ for $\sin^2B$ the terms reduce to 0.

Also solved by C. H. Brown, Richard Yates, G. E. Cadman and Adolph Pierrot, of Chicago, Ill.; Wm. Hoover, Ph. D., Athens, Ohio; E. L. Sherwood, Pittsburg, Pa.; S. A. Cory, Hiteman, Iowa; Nelson L. Roray, Palmyra, N. J.; M. E. Graber, Tiffin, Ohio; A. H. Holmes, Brunswick, Maine; G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va.; F. D. Posey, A. B., San Mateo, Cal.; Christian Hornung, Tiffin, O.; J. Scheffer. Hagerstown, Md.; and the Proposer.

PROBLEMS FOR SOLUTION.

ALGEBRA.

193. Errata. For $\sum \frac{a_i^2}{h_i^2}$ read $\sum \frac{a_i^2}{h_i}$.

195. Proposed by W. J. GREENSTREET, A. M., Editor of the Mathematical Gazette, Stroud, England.

Prove that when n is a positive integer,

$$\sum_{r=1}^{r=n} (-1)^r \, {}_nC_r \, 2^{n-r} \, r^2 = n^2 - 2n.$$

196. Proposed by L. E. NEWCOMB, Los Gatos, California.

Find the rth term of $\left(x - \frac{1}{x}\right)^n \equiv z^n$ in terms of z.

197. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Solve $(18)^{4(2-x)} = (54\sqrt{2})^{3x-2}$.

GEOMETRY.

221. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Construct a right triangle with given hypotenuse h, and having an inscribed square of side 12 with a side lying along the hypotenuse. Show further that the minimum value of h is 36, the triangle being then isosceles.

222. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

At the ends of a focal chord CC' of a parabola are drawn the normal chords CD, $C'D'$. Prove that DD' is parallel to CC' and equal to three times its length.

223. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Find a point C in a given line AB, so the lines joining C to the angular points of a triangle PQR coplanar with the given line may cut off on any line parallel to the given line and lying in the same plane two equal segments.

CALCULUS.

176. Proposed by B. F. FINKEL, A. M., M. Sc., 204 St. Marks Square, Philadelphia, Pa.

Show by any method, Riemann's excepted, that

$$\int_0^\infty e^{-x^2} \cos \frac{b^2}{x^2} dx = \tfrac{1}{2}\sqrt{(\pi)} e^{-b\sqrt{2}} \cos b\sqrt{2}.$$

177. Proposed by O. W. ANTHONY, Head of Mathematical Department, DeWitt Clinton High School, New York City.

Find the volume of the minimum cone which can be circumscribed about a hemisphere.

MECHANICS.

167. Proposed by EDWIN S. CRAWLEY, Ph. D., Head Professor of Mathematics in the University of Pennsylvania.

An anchor ring or torus is produced by the revolution of a circle of radius r, the center of the revolving circle describing a circle of radius R. A quadrant of the torus is cut by two planes through the center of the ring perpendicular to each other and perpendicular to the plane of revolution. Determine the limiting value of the ratio $R:r$, so that when the quadrant thus formed is placed with one of its ends in coincidence with a horizontal plane it will rest in that position.

AVERAGE AND PROBABILITY.

153. Proposed by J. E. SANDERS, Hackney, Ohio.

Three circles, radii $c=1/6a$, are at random within the circumference of a given circle, radius a. Find the chance that (1) all three intersect, (2) two only intersect, (3) none intersect.

NOTES.

Professor Glover of the University of Michigan is giving a two-hour course on Annuities and Insurance.

Associations of Mathematics Teachers have been organized in the States of Kansas and Washington. The former has adopted *School Mathematics* as its official organ.

A Congress of Mathematicians will be held at Heidelberg, Germany, in July to celebrate the hundredth anniversary of Jacobi's birthday. An International Congress of Mathematicians is to be held at St. Louis during the week beginning September 19, 1904.

The current number of the *Transactions of the American Mathematical Society* (January, 1904) contains the following articles: The subgroups of order a power of 2 of the simple quinary orthogonal group in the Galois field of order $p^n=8l\pm3$, by L. E. Dickson; On certain invariants of two triangles, by J. G. Hun; Isothermal systems of geodesics, by E. Kasner; Zur Gruppentheorie mit Anwendungen auf die Theorie der linearen homogenen Differentialgleichungen, by Alfred Loewy; On the group of the sign $(0, 3; 2, 4, \infty)$ and the functions belonging to it, by J. W. Young; On the definition of reducible hypercomplex number systems, by Saul Epsteen; A simple proof of a theorem in the calculus of variations, by E. Goursat.

On February 22nd, at Columbus, Ohio, met a conference in which a very large number of the educational institutions of the State were represented, to form a State organization in the interests of pure science. Dr. George Bruce Halsted, now of Kenyon College, was called upon to preside, and the name chosen was the "Association of Ohio Teachers of Mathematics." The meeting for finally perfecting the organization and adopting the constitution will be held at the end of March or near the beginning of April. At this meeting there will also be a highly important program of papers and discussions. F.

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SPHERICAL GEOMETRY.

By EDWIN BIDWELL WILSON.

LECTURE IV. THE PARTITION OF THE SURFACE.

The theorems which have been derived in the foregoing lectures have been concerned with that which obtains at points of the surface or along lines on the surface. Little has been said of the characteristics of the surface taken as a whole. It will be remembered that in the definition of a triangle a caution not to regard the triangle as a portion of the surface was especially mentioned. It would be equally improper to speak of the interior or exterior of the triangle: for no demonstration or axiom has yet been given to show that the triangle has properties such as are implied by the use of these words. With this deficiency is closely connected (though it might not appear so at first) the rudimentary state of our knowledge of angles, due as has been seen to the fact that we are not assured of any natural arrangement among the directions issuing from a point. This renders it difficult if not impossible to proceed to right angles and the relations of symmetry. To set aside both these limitations we shall now investigate the statement: A line divides the surface of the sphere into two separate parts.

Describe a line l on the surface. Relative to this line two points A and B may have one of two positions. If the line joining A and B cuts the line l in L and M both of which lie within or without the segment AB the two points A and B may be said to lie on the same side of the line l: but if the points L and M are so situated that one lies within the segment AB and the other without that segment, then it would be natural to say that A and B lie on opposite sides of the line l. Thus the points of the surface may be assorted into three classes of which the first contains all the points on the line l, the second all the points